Beam behavior and its measurement in a solenoidal cooling channel

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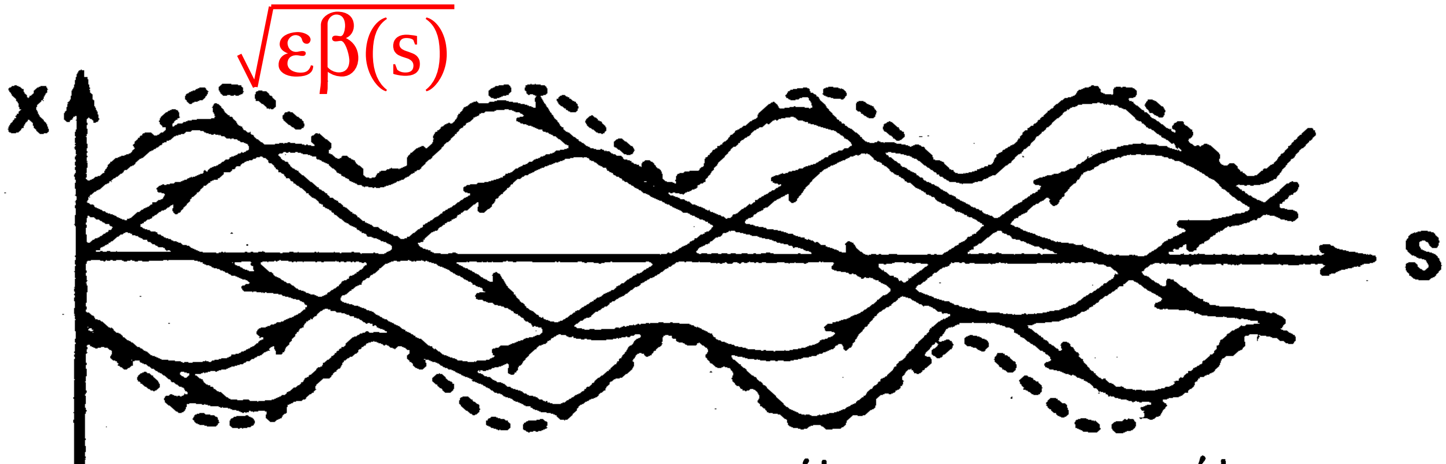
Contents

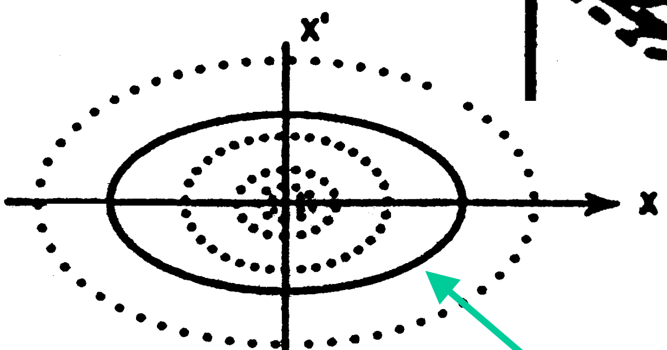
- Basic beam concepts and Courant-Snyder theory (review)
- Beam behavior in a periodic solenoidal channel without cooling
- Cooling behavior

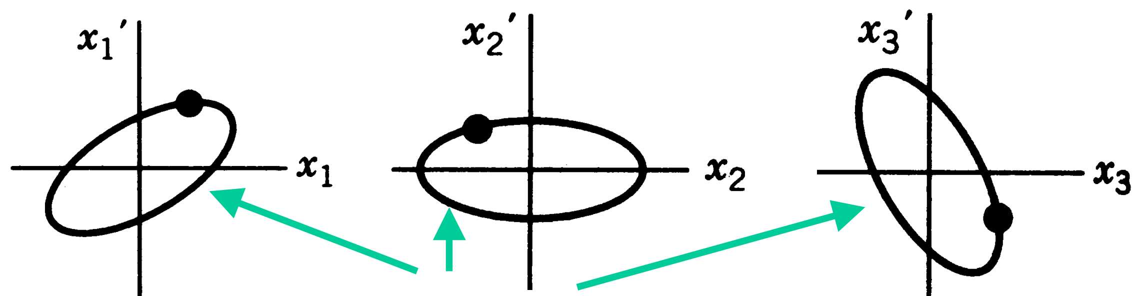
Transverse dynamics of a monochromatic beam in a straight periodic solenoidal cooling channel

How to describe beam hehavior?

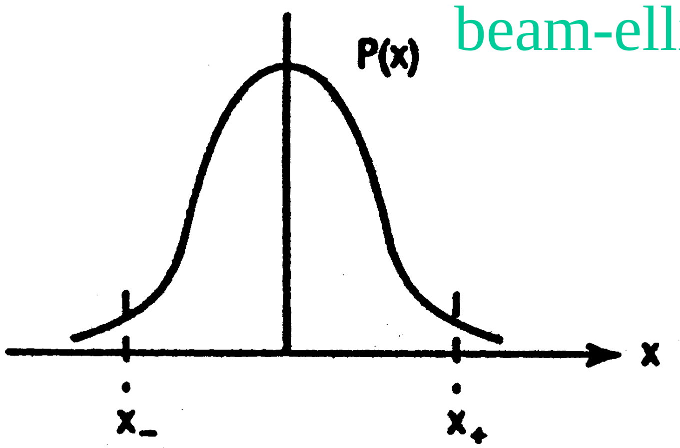


2D phase-space



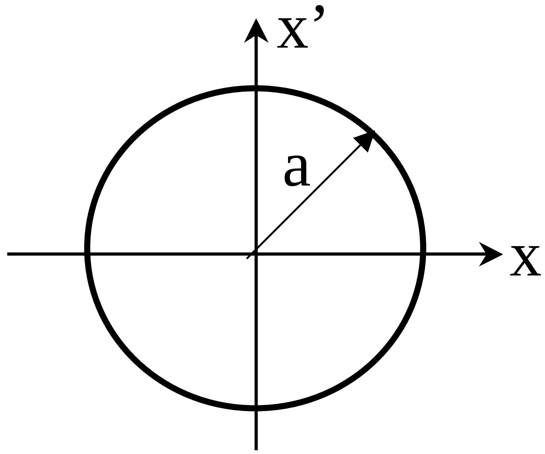
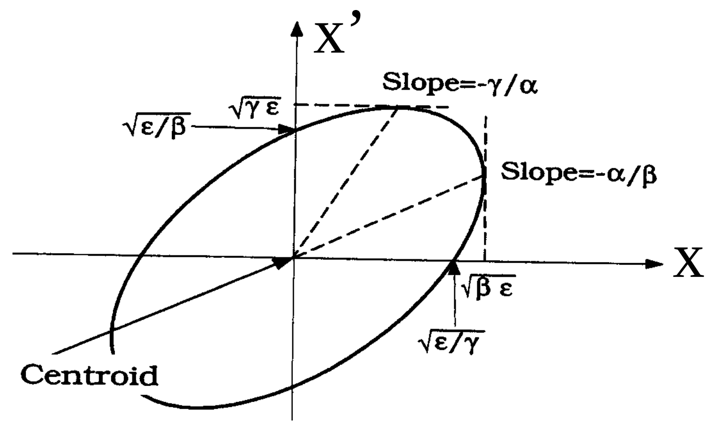


beam-ellipse vs. machine-ellipse



- Phase-space distribution
- Beam emittance ε
- Beam envelope-function β(s)

Courant-Snyder theory

	Harmonic oscillation	Anharmonic oscillation
		
❑Restoring force:	$-kx$	$-k(s)x$
❑Equation of motion:	$x'' + kx = 0$	$x'' + k(s)x = 0$
❑Hamiltonian	$H = \frac{1}{2}(P_x^2 + kx^2)$	$H = \frac{1}{2}(P_x^2 + k(s)x^2)$
❑Solution	$x = a\cos\sqrt{k}s, P_x = -a\sin\sqrt{k}s$	$x = \sqrt{I\beta(s)}\cos(\psi(s)), P_x = \cdots$
❑Invariant	$I = a^2 = x^2 + P_x^2$	$I = \gamma(s)x^2 + 2\alpha(s)xP_x + \beta(s)P_x^2$

❑emittance

❑Floquet transformation

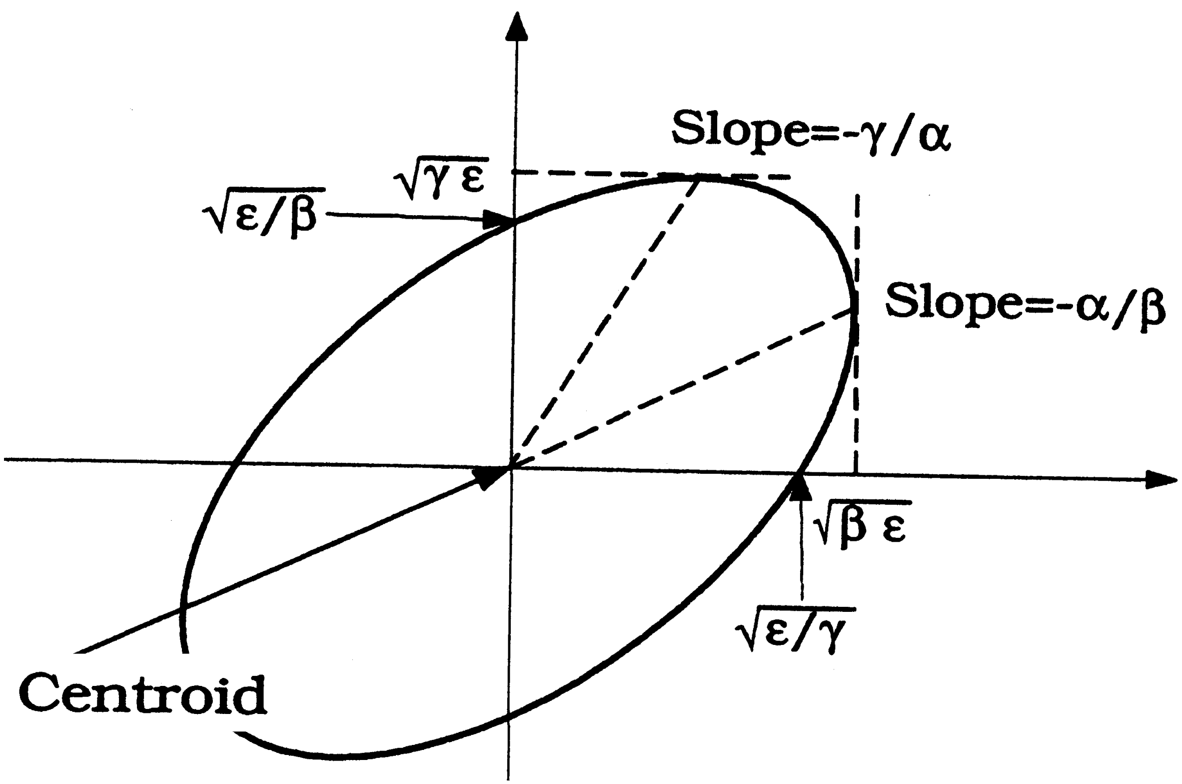
$$\begin{bmatrix} \sqrt{\beta(s)} & 0 \\ -\frac{\alpha(s)}{\beta(s)} & \frac{1}{\sqrt{\beta(s)}} \end{bmatrix}$$

Machine-ellipse and Twiss parameters

Courant-Snyder invariant

$$H = \frac{1}{2}\left(P_x^2 + k(s)x^2\right)$$

$$I = \gamma(s)x^2 + 2\alpha(s)xP_x + \beta(s)P_x^2$$



$$\beta' = -2\alpha$$

$$\alpha' = \kappa(s)^2\beta - \gamma$$

$$\gamma = (1+\alpha^2)/\beta$$

$$\kappa \equiv qB(s)/2P_s$$

Beam phase-space distribution

Gaussian beam

- 1D $\rho(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{x^2}{2\sigma_x^2}}$
- 2D $\rho(x, P_x) = \frac{1}{2\pi\epsilon_x} e^{-\frac{\gamma x^2 + 2\alpha x P_x + \beta P_x^2}{2\epsilon_x}}$
- 4D $\rho(X) = \frac{1}{(2\pi)^2\sqrt{\det(\Sigma)}} e^{-\frac{1}{2}X^T\Sigma^{-1}X}$

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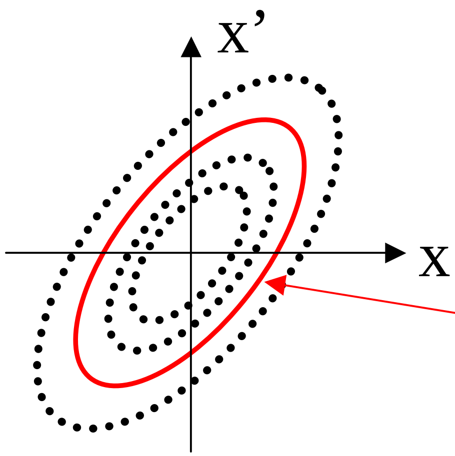
beam moments

$$\Sigma = \begin{bmatrix} \langle x^2\rangle & \langle xP_x\rangle & \langle xy\rangle & \langle xP_y\rangle \\ \langle xP_x\rangle & \langle P_x^2\rangle & \langle yP_x\rangle & \langle P_xP_y\rangle \\ \langle xy\rangle & \langle yP_x\rangle & \langle y^2\rangle & \langle yP_y\rangle \\ \langle xP_y\rangle & \langle P_xP_y\rangle & \langle yP_y\rangle & \langle P_y^2\rangle \end{bmatrix}$$

2D emittance $\epsilon_x \equiv \sqrt{\langle x^2\rangle\langle P_x^2\rangle - \langle xP_x\rangle^2}$

Beam-ellipse and equilibrium (matched) distribution



equi-density contours

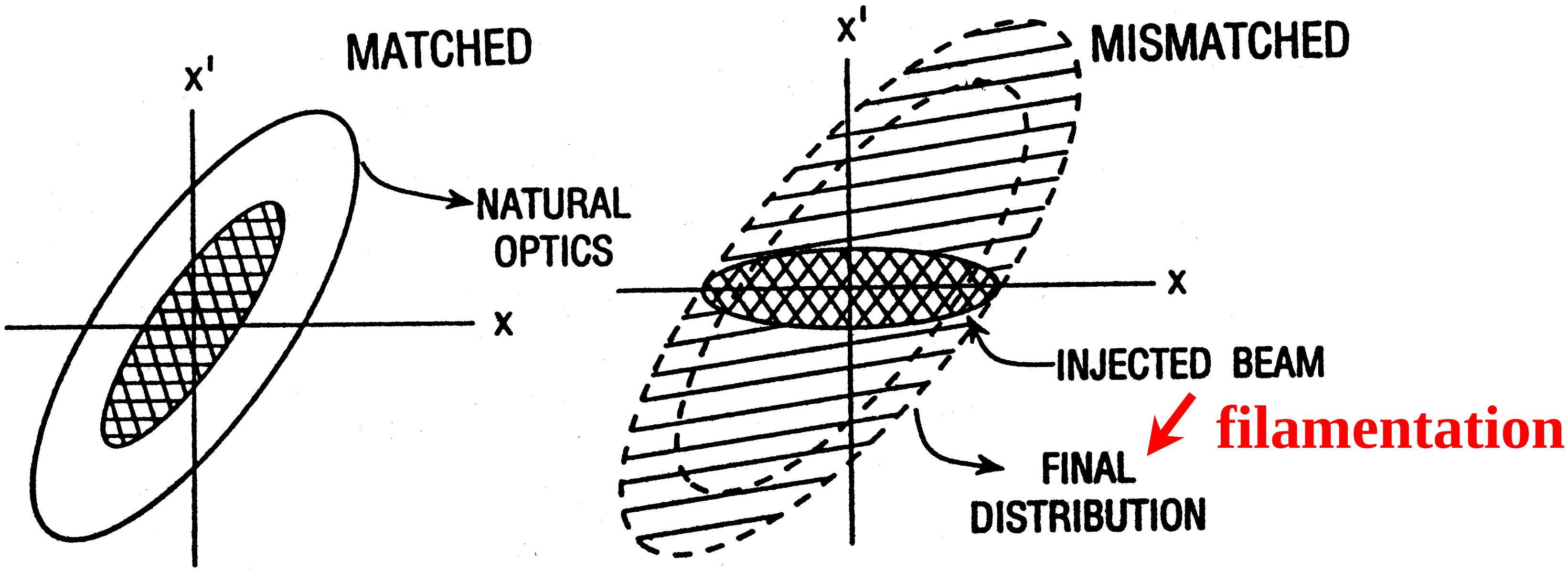
any beam — matched beam

$$\begin{bmatrix}\langle x^2\rangle & \langle xP_x\rangle \\ \langle xP_x\rangle & \langle P_x^2\rangle\end{bmatrix} \equiv \epsilon \begin{bmatrix}\beta & -\alpha \\ -\alpha & \gamma\end{bmatrix} = \epsilon \begin{bmatrix}\hat{\beta} & -\hat{\alpha} \\ -\hat{\alpha} & \hat{\gamma}\end{bmatrix}$$

beam ellipse — machine-ellipse

Since individual particle moves on the machine-ellipse,
beam-ellipse must match (be the same as) the machine-ellipse
for a beam to reach equilibrium distribution---matched beam.

Emittance degradation of mismatched beam



$$\beta_{mag} \equiv \frac{\bar{\beta}_x + \bar{\gamma}_x}{2} = -\alpha_x(0)\hat{\alpha}(0) + \frac{\beta_x(0)\hat{\beta}(0) + \gamma_x(0)\hat{\gamma}(0)}{2}$$

$$= \frac{1}{2}\left\{\frac{\beta_x}{\hat{\beta}} + \frac{\hat{\beta}}{\beta_x} + \left(\sqrt{\frac{\beta_x}{\hat{\beta}}}\hat{\alpha} - \sqrt{\frac{\hat{\beta}}{\beta_x}}\alpha_x\right)^2\right\}_{s=s_0}$$

Solenoidal focusing field

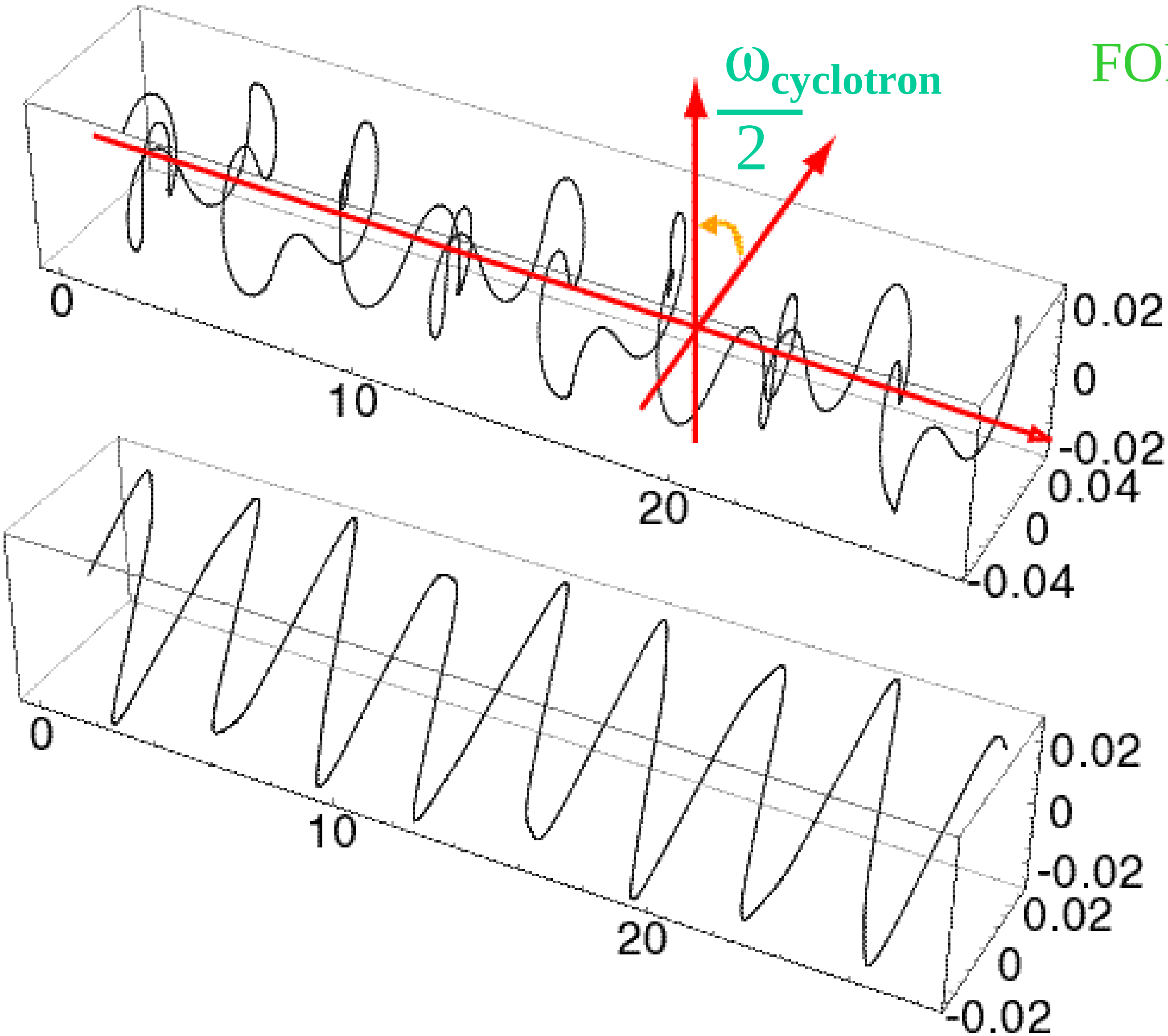
$$\mathbf{B}(x, y, s) = B(s)\mathbf{e}_z - \frac{1}{2}B'(s)(x\mathbf{e}_x + y\mathbf{e}_y)$$

+ nonlinear terms

- Completely determined by the on-axis field due to symmetry
- Be prepared to give this up if dealing with field errors

Larmor rotating frame

FOFO channel of 3.4 T



In lab frame

In Lamor frame

Single particle dynamics

In Lamor frame, x & y decoupled and linear dynamics is governed by Hill's equation as in quadrupole channels

$$\frac{d^2x(s)}{ds^2} + \kappa(s)^2 x(s) = 0$$

In lab frame $$\frac{d}{ds}p_s\frac{d\mathbf{x}}{ds} = -qB(s)\mathbf{e}_s\times\frac{d\mathbf{x}}{ds} - \frac{q}{2}\frac{dB}{ds}\mathbf{e}_s\times\mathbf{x} - \eta p_s\frac{d\mathbf{x}}{ds} + p_s\mathbf{u}$$

In Lamor $$\frac{d}{ds}p_s\frac{d\mathbf{x}_R}{ds} = -p_s\kappa^2\mathbf{x}_R - p_s\eta\left(\frac{d\mathbf{x}_R}{ds} - \kappa\mathbf{e}_s\times\mathbf{x}_R\right) + p_s\mathbf{u}_R$$

In lab frame $$H = \frac{1}{2}\left(P_x^2 + k^2x^2\right) + \frac{1}{2}\left(P_y^2 + k^2y^2\right) - k\left(xP_y - yP_x\right)$$

In Lamor $$\tilde{H} = \frac{1}{2}\left(\tilde{P}_x^2 + k^2\tilde{x}^2\right) + \frac{1}{2}\left(\tilde{P}_y^2 + k^2\tilde{y}^2\right)$$ no x & y coupling!

Beam moment parameterization (Larmor)

x $\{\langle\tilde{x}^2\rangle, \langle\tilde{x}\tilde{P}_x\rangle, \langle\tilde{P}_x^2\rangle\} = \epsilon_x\{\hat{\beta}(s), -\hat{\alpha}(s), \hat{\gamma}(s)\}$

y $\{\langle\tilde{y}^2\rangle, \langle\tilde{y}\tilde{P}_y\rangle, \langle\tilde{P}_y^2\rangle\} = \epsilon_y\{\hat{\beta}(s), -\hat{\alpha}(s), \hat{\gamma}(s)\}$

a $\{\langle\tilde{x}\tilde{y}\rangle, \frac{\langle\tilde{x}\tilde{P}_y\rangle + \langle\tilde{y}\tilde{P}_x\rangle}{2}, \langle\tilde{P}_x\tilde{P}_y\rangle\} = \epsilon_{xy}\{\hat{\beta}, -\hat{\alpha}, \hat{\gamma}\}$

s $\langle\tilde{x}\tilde{P}_y\rangle - \langle\tilde{y}\tilde{P}_x\rangle = \langle L_z\rangle \equiv L$

s $\{\langle\tilde{x}^2\rangle+\langle\tilde{y}^2\rangle, \langle\tilde{x}\tilde{P}_x\rangle+\langle\tilde{y}\tilde{P}_y\rangle, \langle\tilde{P}_x^2\rangle+\langle\tilde{P}_y^2\rangle\} = 2\epsilon_s\{\hat{\beta}, -\hat{\alpha}, \hat{\gamma}\}$

a $\{\langle\tilde{x}^2\rangle-\langle\tilde{y}^2\rangle, \langle\tilde{x}\tilde{P}_x\rangle-\langle\tilde{y}\tilde{P}_y\rangle, \langle\tilde{P}_x^2\rangle-\langle\tilde{P}_y^2\rangle\} = 2\epsilon_a\{\hat{\beta}, -\hat{\alpha}, \hat{\gamma}\}$

s: rotationally symmetric a: asymmetric

4 beam parameters

$$\epsilon_x \equiv \sqrt{\langle \tilde{x}^2 \rangle \langle \tilde{P}_x^2 \rangle - \langle \tilde{x}\tilde{P}_x \rangle^2} = \frac{1}{2}\langle I_x \rangle$$

$$\epsilon_y \equiv \sqrt{\langle \tilde{y}^2 \rangle \langle \tilde{P}_y^2 \rangle - \langle \tilde{y}\tilde{P}_y \rangle^2} = \frac{1}{2}\langle I_y \rangle$$

$$\epsilon_{xy} \equiv \sqrt{\langle \tilde{x}\tilde{y} \rangle \langle \tilde{P}_x \tilde{P}_y \rangle - \langle \frac{\tilde{x}\tilde{P}_y + \tilde{y}\tilde{P}_x}{2} \rangle^2} = \frac{1}{2}\langle I_{xy} \rangle$$

$$L \equiv \langle \tilde{x}\tilde{P}_y \rangle - \langle \tilde{y}\tilde{P}_x \rangle = \langle L_z \rangle$$

Courant-Snyder type single-particle invariants

$$I_x = \hat{\gamma}(s)\tilde{x}^2 + 2\hat{\alpha}(s)\tilde{x}\tilde{P}_x + \hat{\beta}(s)\tilde{P}_x^2\,, \text{ etc}$$

Moments of equilibrium beam (Lab)

$$\{\langle x^2\rangle, \langle xP_x\rangle, \langle P_x^2\rangle\} = (\epsilon_s + \sqrt{\epsilon_a^2 + \epsilon_{xy}^2}\sin\theta)\{\hat{\beta}, -\hat{\alpha}, \hat{\gamma}\}$$

$$\{\langle y^2\rangle, \langle yP_y\rangle, \langle P_y^2\rangle\} = (\epsilon_s - \sqrt{\epsilon_a^2 + \epsilon_{xy}^2}\sin\theta)\{\hat{\beta}, -\hat{\alpha}, \hat{\gamma}\}$$

$$\{\langle xy\rangle, \langle P_x P_y\rangle\} = \sqrt{\epsilon_a^2 + \epsilon_{xy}^2}\cos\theta\,\{\hat{\beta}(s), \hat{\gamma}(s)\}$$

$$\langle xP_y\rangle = \quad L/2 - \sqrt{\epsilon_a^2 + \epsilon_{xy}^2}\cos\theta\,\hat{\alpha}(s)$$

$$\langle yP_x\rangle = -L/2 - \sqrt{\epsilon_a^2 + \epsilon_{xy}^2}\cos\theta\,\hat{\alpha}(s)$$

$$\theta(s) = 2\theta_L - \arctan(\epsilon_a/\epsilon_{xy}), \quad \theta_L(s) = -\frac{q}{2P_s}\int_0^s B(\bar{s})d\bar{s}$$

Measurement of emittances

$$\epsilon_s = \left(\langle x^2\rangle + \langle y^2\rangle\right)/2\hat{\beta}(s),$$

$$\epsilon_a^2 + \epsilon_{xy}^2 = \left[\left(\langle x^2\rangle - \langle y^2\rangle\right)^2 + 4\langle xy\rangle^2\right] \Big/ 4\hat{\beta}^2(s)$$

$$\frac{\epsilon_a}{\epsilon_{xy}} = \tan\left[2\theta_L - \cot^{-1}\frac{2\langle xy\rangle}{\langle x^2\rangle - \langle y^2\rangle}\right]$$

- Reply on measurement of beam profile instead of position.
- Windows in a muon cooling channel provide unique opportunity for such measurements!!

Envelope equations with cooling

$$\frac{d}{ds}\begin{pmatrix}\epsilon_s \\ L\end{pmatrix} = -\begin{pmatrix}\eta & -\eta\kappa\beta \\ -\eta\kappa\beta & \eta\end{pmatrix}\begin{pmatrix}\epsilon_s \\ L\end{pmatrix} + \begin{pmatrix}\beta\chi \\ 0\end{pmatrix}$$

$$\frac{d\epsilon_a}{ds} = -\eta\,\epsilon_a\;, \qquad \frac{d\epsilon_{xy}}{ds} = -\eta\,\epsilon_{xy}$$

$$\frac{d\beta}{ds} + 2\alpha = \eta\beta - \frac{\eta\kappa L + \chi}{\epsilon}\beta^2$$

$$\frac{d\alpha}{ds} + \frac{1+\alpha^2}{\beta} - \kappa^2\beta = -\frac{\eta\kappa L + \chi}{\epsilon}\beta\,\alpha.$$

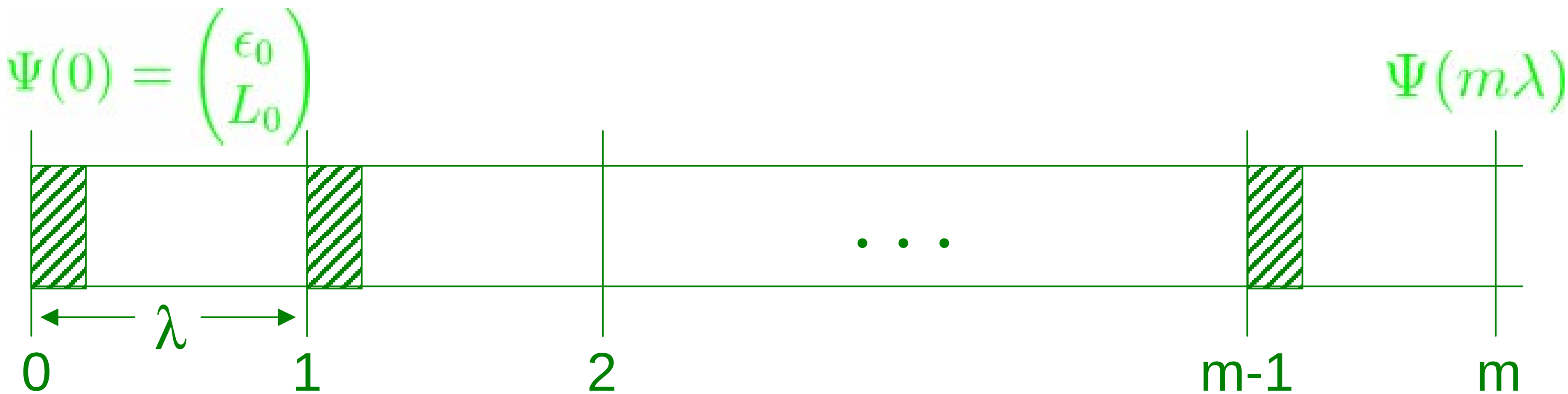
$$\kappa(s) = \frac{qB(s)}{2p_s}, \quad \eta = \frac{1}{p_s v}\frac{dE}{ds}\bigg|_{loss} = \frac{1}{p_s}\frac{dp}{ds}\bigg|_{loss}, \quad \chi(s) = \left(\frac{13.6\,\mathrm{MeV}}{\mathrm{pv}}\right)^2\frac{1}{L_{Rad}}$$

In periodic cooling channels

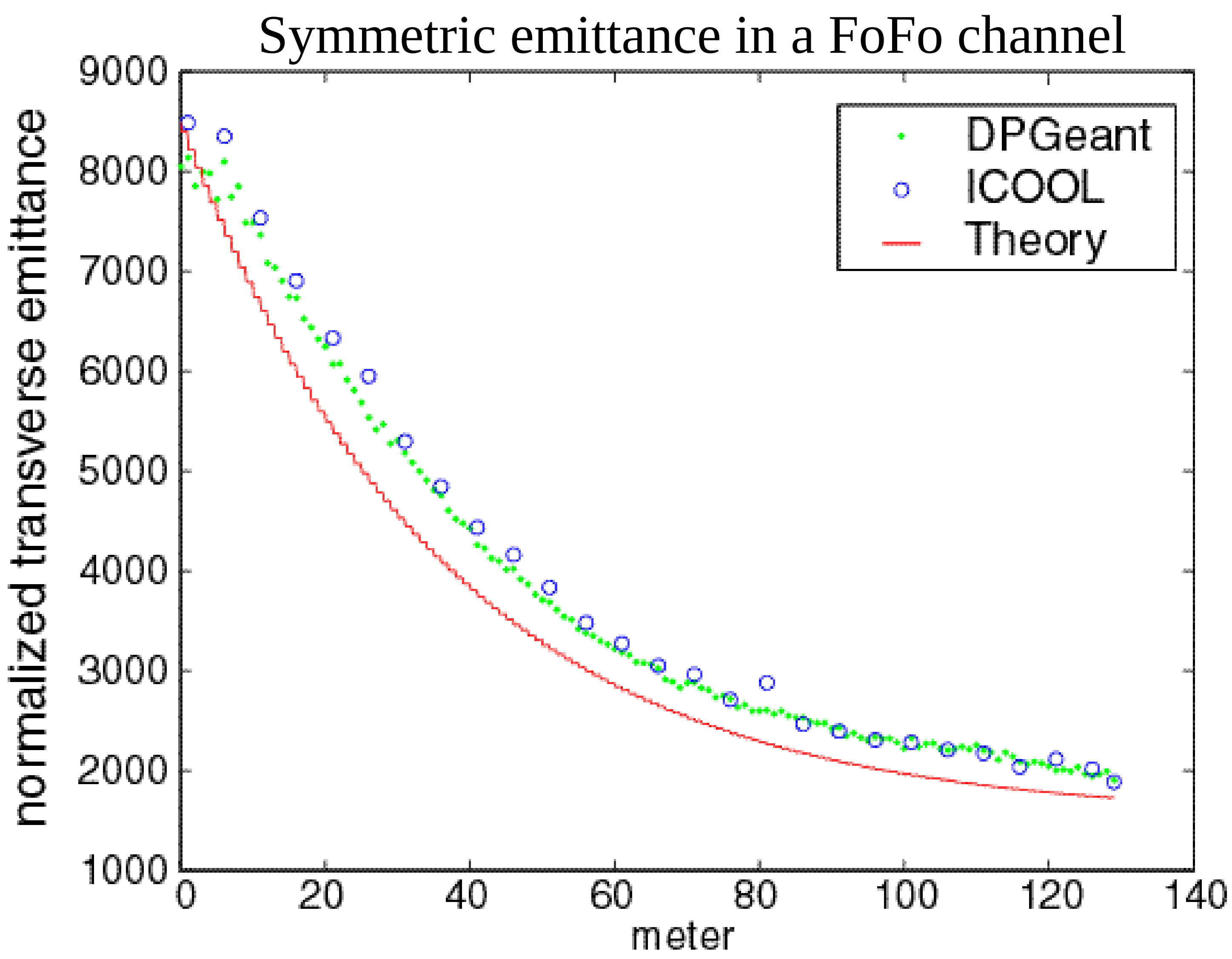
Damped initial

$$\Psi(m\lambda) = \underbrace{e^{-m\Gamma(\lambda)}\Psi(0)} + \left\{1 + e^{-\Gamma(\lambda)} + \cdots + e^{-(m-1)\Gamma(\lambda)}\right\} e^{-\Gamma(\lambda)}\int_0^\lambda ds\, e^{\Gamma(s)} \begin{pmatrix} \beta\chi \\ 0 \end{pmatrix}$$

Damping of heating in previous periods **Heating in a period**



Cooling performance



Cooling behavior summary

1) Incoming beam quickly matches into a periodic cooling channel via filamentation and particle lose.
2) The asymmetric part of the matched beam will be cooled away exponentially without heating!
3) The symmetric emittance and angular momentum will be cooled by ionization energy loss while heated by multiple scattering.
4) The final equilibrium state is a round beam with only symmetric emittance determined by the balance of cooling and heating. (may have net angular momentum depends on channel design)